Matrix Multiplication, Inverse Matrices, and Determinants

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Overview

- Review last lecture
  - Background for matrix multiplication
  - General rule and examples of matrix multiplication
- Definition, computation and use of determinants
- Use of determinants in computing matrix inverses
- Example problems

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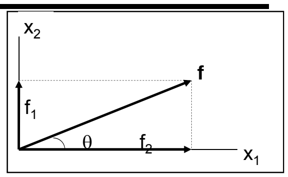
Review last lecture

- Vector has a magnitude and a direction
  - Can represent by components



- **f·dx = |f||dx|**cos(θ)
- $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ or $\mathbf{f} = [f_1 \quad f_2 \quad f_3]$
- Unit vectors **i** = [1 0 0], **j** = [0 1 0], **k** = [0 0 1] in x, y, z directions
- **Dot product: f·dx** = $f_1dx_1 + f_2dx_2 + f_3dx_3$

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Review Matrix Basics

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} \end{bmatrix}$$

- Array of numbers with n rows and m columns
- Components are $a_{(row)(column)}$
- Size of matrix (n x m) is number of rows and columns
- Square matrix: m = n

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Review Diagonal Matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_n \end{bmatrix}$$

- The diagonal matrix **A** is a square matrix with nonzero components only on the principal diagonal
- Components of **A** are $a_i\delta_{ij}$, where δ_{ij} is the Kroenecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

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Review Matrix Operations

- Can add or subtract matrices if they are the same size
  - **C** = **A** ± **B** only valid if **A**, **B**, and **C** have the same size (rows and columns)
  - Components of **C**, $c_{ij} = a_{ij} \pm b_{ij}$
- Multiplication by a scalar: **C** = x**A**
  - **C** and **A** have the same size (rows and columns)
  - Components of **C** = x**A**, $c_{ij} = xa_{ij}$

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Review Null (**0**)/Unit (**I**) Matrices

- For any matrix, **A**, **A** + **0** = **0** + **A** = **A; IA** = **AI** = **A** and **0 A** = **A 0** = **0**
- The unit (or identity) matrix is a square matrix; the null matrix need not be square

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & & & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

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Unit and Null Matrix sizes

- Given that **A** is an (n x m) matrix
- The **I** in **IA** = **A** is (n x n)
- The **I** in **AI** = **A** is (m x m)
- The **0** in **0** + **A** = **A** or **A** + **0** = **A** is (n x m)
- The **0** in **A0** is (m x arbitrary size)
  - The product **A0** is also a zero matrix whose size is (n x null matrix columns)
- The **0** in **0A** is (arbitrary size x n)
  - The product **0A** is also a zero matrix whose size is (null matrix rows x m)

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Review Matrix Transpose

- Transpose of **A** denoted as $\mathbf{A}^T$ (or **A**')
- Reverse rows and columns; for **B** = $\mathbf{A}^T$
  - $b_{ij} = a_{ji}$
  - If **A** is (n x m), **B** = $\mathbf{A}^T$ is (m by n)

$$\mathbf{A} = \begin{bmatrix} 3 & 12 & -6 \\ 14 & -2 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 3 & 14 \\ 12 & -2 \\ -6 & 0 \end{bmatrix}$$

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Review Row/Column Vectors

- Matrices with only one row or only one column are called row or column vectors (or matrices)

$$\mathbf{r} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & \cdots & r_{1m} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & \cdots & r_m \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ \vdots \\ \vdots \\ c_{n1} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}$$

- Single subscript is usual notation, but implied "1" for single row or column is important for two-subscript formulas

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Review Multiplying Matrices

- For matrix multiplication, **C** = **AB**
  - **A** has n rows and p columns
  - **B** has p rows and m columns
  - **C** has n rows and m columns

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad (i = 1, n;\ j = 1, m)$$

- For **C** = **AB**, we get c_{ij} by adding products of terms in row i of **A** (left matrix) by terms in column j of **B** (right matrix)
- $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} + \ldots$
- In general, **AB** ≠ **BA**
- "Premultiply" by matrix on left and "post-multiply" by matrix on right

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Matrix Multiplication Example

- For matrix multiplication, **C** = **AB**
  - **A** has n rows and p columns
  - **B** has p rows and m columns
  - **C** has n rows and m columns

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad (i = 1, n;\ j = 1, m)$$

- Example

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -6 \\ 4 & -2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 6 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3(3) + 0(1) - 6(6) & 3(4) + 0(2) - 6(1) \\ 4(3) - 2(1) + 0(6) & 4(4) - 2(2) + 0(1) \end{bmatrix} = \begin{bmatrix} -27 & 6 \\ 10 & 12 \end{bmatrix}$$

- Can we find **BA**? What is its size? **3 x 3**

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Matrix Multiplication Exercise

- Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- Can you find **AB**, **BA**, neither or both?
- We can find **AB**, because **A** has three columns and **B** has three rows
- We cannot find **BA** because **B** has four columns and **A** has three rows

Matrix Multiplication Exercise II

- What is the size of **C** = **AB**?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- **C** = **AB** has three rows (like **A**) and four columns (like **B**)
- What is c_{11}?
- $c_{11} = (1)(0) + (2)(1) + (3)(2) = 8$

Matrix Multiplication Exercise III

- Find c_{11}, c_{12}, c_{13}, and c_{14} in **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- $c_{11} = (1)(0) + (2)(1) + (3)(2) = 8$
- $c_{12} = (1)(2) + (2)(3) + (3)(1) = 11$
- $c_{13} = (1)(-2) + (2)(1) + (3)(-3) = -9$
- $c_{14} = (1)(0) + (2)(0) + (3)(1) = 3$

Matrix Multiplication Exercise IV

- Find c_{21}, c_{22}, c_{23}, and c_{24} in **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- $c_{21} = (0)(0) + (-1)(1) + (4)(2) = 7$
- $c_{22} = (0)(2) + (-1)(3) + (4)(1) = 1$
- $c_{23} = (0)(-2) + (-1)(1) + (4)(-3) = -13$
- $c_{24} = (0)(0) + (-1)(0) + (4)(1) = 4$

Matrix Multiplication Exercise V

- Find c_{31}, c_{32}, c_{33}, and c_{34} in **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- $c_{31} = (1)(0) + (1)(1) + (0)(2) = 1$
- $c_{32} = (1)(2) + (1)(3) + (0)(1) = 5$
- $c_{33} = (1)(-2) + (1)(1) + (0)(-3) = -1$
- $c_{34} = (1)(0) + (1)(0) + (0)(1) = 0$

Matrix Multiplication Exercise VI

- Solution for matrix product **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 8 & 11 & -9 & 3 \\ 7 & 1 & -13 & 4 \\ 1 & 5 & -1 & 0 \end{bmatrix}$$

Coordinate transformations

- Recall previous equations

$y_1 = a_{11}x_1 + a_{12}x_2$ $z_1 = b_{11}y_1 + b_{12}y_2$

$y_2 = a_{21}x_1 + a_{22}x_2$ $z_2 = b_{21}y_1 + b_{22}y_2$

- Define matrices so that **y** = **Ax** and **z** = **By**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

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Coordinate transformations II

- Show that matrix definitions give transformation results

$y_1 = a_{11}x_1 + a_{12}x_2$ $z_1 = b_{11}y_1 + b_{12}y_2$

$y_2 = a_{21}x_1 + a_{22}x_2$ $z_2 = b_{21}y_1 + b_{22}y_2$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{By} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_{11}y_1 + b_{12}y_2 \\ b_{21}y_1 + b_{22}y_2 \end{bmatrix}$$

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Coordinate Transformations III

- From matrix equations **y** = **Ax** and **z** = **By**, we have **z** = **BAx** = **Cx** with **C** = **BA**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{Cx} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_{11}x_1 + c_{12}x_2 \\ c_{21}x_1 + c_{22}x_2 \end{bmatrix}$$

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Determinants

- Looks like a matrix but isn't a matrix
- A square array of numbers with a rule for computing a single value for the array

Example at right shows calculation of Det(**A**), the determinant of 3 x 3 matrix **A**

$$Det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}$$

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General Determinant

- The value of an n by n determinant is found by taking a sum of n! terms
- Each term in the sum has
  - a product of n determinant elements, a_{ij}
  - an element from each row and an element from each column: $a_{\alpha 1}a_{\beta 2}a_{\gamma 3}a_{\delta 4}\ldots$
    - Subscripts $\alpha\beta\gamma\delta\ldots$ are a permutation of 1234…
- Terms are positive or negative if the subscript permutation is even or odd
- Practical calculation formulas follow

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More Determinants

- Useful in obtaining algebraic expressions for matrix operations, but not useful for numerical computation

$$Det\ \mathbf{A} = Det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

- An n x n determinant has n^2 Minors, M_{ij}, obtained by deleting row i and column j
- Cofactors, $C_{ij} = (-1)^{i+j}M_{ij}$ used in general expressions for determinants

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General Rule for Determinants

- Any size determinant can be evaluated by any of the following equations

$$Det\,\mathbf{A} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij} = \sum_{i=1}^{n}a_{ij}C_{ij} = \sum_{j=1}^{n}a_{ij}C_{ij}$$

- Can pick ***any*** row ***or any*** column
- Choose row or column with most zeros to simplify calculations
- Can apply equation recursively; evaluate a 5 x 5 determinant as a sum of 4 x 4 determinants then get 4 x 4's in terms of 3 x 3's

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Determinant Behavior

- A determinant is zero if any row or any column contains all zeros.
- If one row or one column of a determinant is multiplied by a constant, k, the value of the determinant is multiplied by the same constant.
  - Note the implication for matrices: if a matrix is multiplied by a constant, k, then each matrix element is multiplied by k. If **A** is an n x n matrix, Det(k**A**) = k^nDet(**A**).

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Determinant Behavior II

- If one row (or one column) of a determinant is replaced by a linear combination of that row (or column) with another row (or column), the value of the determinant is not changed.
- If two rows (or two columns) of a determinant are linearly dependent the value of the determinant is zero.

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Determinant Behavior III

- The determinant of the product of two matrices, **A** and **B** is the product of the determinants of the individual matrices: Det(**AB**) = Det(**A**) Det(**B**).
- The determinant of transposed matrix is the same as the determinant of the original matrix: Det($\mathbf{A}^T$) = Det(**A**).

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Example of General Rule

- Get determinant of a 3 x 3 matrix by expansion along last row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$C_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad C_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2}M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad C_{31} = (-1)^{3+1}M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

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Example of General Rule II

- Get determinant of a 3 x 3 matrix

$$a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33} =$$

$$a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} =$$

$$= a_{31}(a_{12}a_{23} - a_{22}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})$$

$$= a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{11}a_{23} + a_{32}a_{21}a_{13} + a_{33}a_{11}a_{22} - a_{33}a_{12}a_{21}$$

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Inverse of a Matrix

- For a ***square*** matrix, **A**, an inverse matrix, $\mathbf{A}^{-1}$ ***may exist*** such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- For the algebraic equation ax = b, $x = a^{-1}b$
- For the matrix equation **Ax** = **b**, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- Just as $x = a^{-1}b$ is not valid if a = 0, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is not valid if $\mathbf{A}^{-1}$ does not exist
  - $\mathbf{A}^{-1}$ does not exist if Det**A** = 0
- The inverse is an important concept in analysis of linear systems, but is not used in computational work

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Formula for Inverse of a Matrix

- Find the components of $\mathbf{B} = \mathbf{A}^{-1}$, b_{ij}, from determinant of **A** and its cofactors

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad b_{ij} = \frac{C_{ji}}{Det(\mathbf{A})} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})}$$

- Use to get algebraic equations for components of inverse matrix
- Matrix computations, if necessary, obtain components by alternative numerical algorithms – more next two weeks

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Inverse of 2 x 2 Matrix

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad b_{ij} = \frac{C_{ji}}{Det(\mathbf{A})} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})}$$

$$b_{11} = (-1)^{1+1}\frac{M_{11}}{Det(\mathbf{A})} = \frac{a_{22}}{Det(\mathbf{A})} \qquad b_{12} = (-1)^{1+2}\frac{M_{21}}{Det(\mathbf{A})} = -\frac{a_{12}}{Det(\mathbf{A})}$$

$$b_{21} = (-1)^{2+1}\frac{M_{12}}{Det(\mathbf{A})} = -\frac{a_{21}}{Det(\mathbf{A})} \qquad b_{22} = (-1)^{2+2}\frac{M_{22}}{Det(\mathbf{A})} = \frac{a_{11}}{Det(\mathbf{A})}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

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Inverse of 3 x 3 Matrix

$$Apply \quad \mathbf{B} = \mathbf{A}^{-1}, \quad b_{ij} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} (a_{22}a_{33} - a_{32}a_{23}) & (a_{32}a_{13} - a_{33}a_{12}) & (a_{12}a_{23} - a_{22}a_{13}) \\ (a_{31}a_{23} - a_{33}a_{21}) & (a_{11}a_{33} - a_{31}a_{13}) & (a_{21}a_{13} - a_{11}a_{23}) \\ (a_{21}a_{32} - a_{31}a_{22}) & (a_{31}a_{12} - a_{11}a_{32}) & (a_{11}a_{22} - a_{21}a_{12}) \end{bmatrix}}{\begin{pmatrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{pmatrix}}$$

$(-1)^{(2+3)}M_{32}$

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Example Problem

- Find $\mathbf{A}^{-1}$ for **A** at right
- Have the following formula for $\mathbf{B} = \mathbf{A}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

$$b_{ij} = \frac{(-1)^{i+j}M_{ji}}{Det(\mathbf{A})} = \frac{C_{ji}}{Det(\mathbf{A})}$$

- General determinant formula: $\Sigma a_{ij}C_{ij}$
- Take sum over third row to simplify calculation of Det **A**

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Example Problem Det A

- Det **A** = $(-1)^{3+1}a_{31}M_{31} + (-1)^{3+2}a_{32}M_{32} + (-1)^{3+3}a_{33}M_{33} + (-1)^{3+4}a_{34}M_{34}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

$$Det\ \mathbf{A} = (-1)^{3+1}(1)\begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix} + 0 + 0 + 0$$

$$Det\ \mathbf{A} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix} = \begin{matrix} (0)(0)(1) + (1)(3)(1) + (2)(2)(0) \\ -(2)(0)(1) - (1)(2)(1) - (0)(3)(0) \end{matrix} = 1$$

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Example Problem III

- Apply: $b_{ij} = (-1)^{i+j}M_{ji}/\text{Det } \mathbf{A}$
- Det **A** = 1 so $b_{ij} = (-1)^{i+j}M_{ji}$
  - Examples b_{12}, b_{43} shown below

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

Remove for M_{21} Remove for M_{34}

$$b_{12} = (-1)^{1+2}M_{21} = -\begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$b_{43} = (-1)^{4+3}M_{34} = -\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} = \begin{matrix} -(3+8+0 \\ -0-0-0) \end{matrix} = -11$$

Example Problem Solution

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -2 & 5 & 1 \\ 3 & 4 & -11 & -2 \end{bmatrix}$$

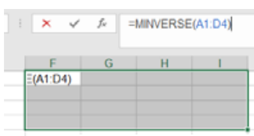
- We can show that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- $(\mathbf{AA}^{-1})_{11} = 1\cdot0 + 0\cdot0 + 2\cdot(-1) + 1\cdot3 = 1$
- $(\mathbf{A}^{-1}\mathbf{A})_{43} = 3\cdot2 + 4\cdot0 + (-11)\cdot0 + (-2)\cdot2 = 0$
- Only 14 left to check

Excel MINVERSE Function

Original Matrix

	A	B	C	D
1	1	0	2	1
2	2	1	0	0
3	1	0	0	0
4	0	2	3	1

Select area for inverse and type MINVERSE formula in formula bar



Press Control+ Shift+Enter

MINVERSE Result

{=MINVERSE(A1:D4)}

F	G	H	I
-1.E-16	-2.E-16	1	1.E-16
2.E-16	1	-2	-2.E-16
-1	-2	5	1
3	4	-11	-2

MINVERSE Result (formula view)

F	G	H	I
=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)
=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)
=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)
=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)	=MINVERSE(A1:D4)